

On perturbations of an ODE with non-Lipschitz coefficients by a small self-similar noise

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Abstract

We study the limit behavior of differential equations with non-Lipschitz coefficients that are perturbed by a small self-similar noise. It is proved that the limiting process is equal to the maximal solution or minimal solution with certain probabilities p_+ and $p_- = 1 - p_+$, respectively. We propose a space-time transformation that reduces the investigation of the original problem to the study of the exact growth rate of a solution to a certain SDE with self-similar noise. This problem is interesting in itself. Moreover, the probabilities p_+ and p_- coincide with probabilities that the solution of the transformed equation converges to $+\infty$ or $-\infty$ as $t \rightarrow \infty$, respectively.

Introduction

We study the limit behavior of the sequence of small-noise stochastic equations

$$X_\varepsilon(t) = \int_0^t (c_+ \mathbb{1}_{X_\varepsilon(s) \geq 0} - c_- \mathbb{1}_{X_\varepsilon(s) < 0}) \operatorname{sign}(X_\varepsilon(s)) |X_\varepsilon(s)|^\alpha ds + \varepsilon B_\beta(t), \quad (1)$$

where $c_\pm > 0$, $\alpha \in (-1, 1)$, $B_\beta(t)$, $t \geq 0$ is a self-similar process with index $\beta > 0$.

Note that the drift $a(x) = (c_+ \mathbb{1}_{x \geq 0} - c_- \mathbb{1}_x) |x|^\alpha \operatorname{sign} x$ does not satisfy the Lipschitz property at 0, and the limit equation

$$X_0(t) = \int_0^t a(X_0(s)) ds, t \geq 0, \quad (2)$$

has two families of solutions

$$X_0^{+, \tau}(t) = \begin{cases} 0, & t \in [0, \tau], \\ (c_+^{\frac{1}{1-\alpha}} (1-\alpha)t)^{\frac{1}{1-\alpha}}, & t \geq \tau; \end{cases} \quad X_0^{-, \tau}(t) = \begin{cases} 0, & t \in [0, \tau], \\ -(c_-^{\frac{1}{1-\alpha}} (1-\alpha)t)^{\frac{1}{1-\alpha}}, & t \geq \tau, \end{cases}$$

where $\tau \in [0, \infty]$ is a parameter.

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Note that if the self-similar process is a Levy stable process or a fractional Brownian motion, then there is a unique solution to the equation (1) under a suitable relation between α and β , see [6, 16, 13]. That's why a limit of $\{X_\varepsilon\}$ as $\varepsilon \rightarrow 0$ can be considered as a natural selection for a solution to (2).

The seminal result on a selection problem was a paper of Bafico and Baldi [1] who considered the case where B_β is a Wiener process. They proved (even for more general form of the drift) that the sequence of distributions of X_ε weakly converges as $\varepsilon \rightarrow 0$ to $p_- \delta_{X^-} + p_+ \delta_{X^+}$, where $p_- + p_+ = 1$ and X^\pm are the maximal and the minimal solutions to the limit equation. In our case $X^\pm(t) = \pm(c_\pm^{\frac{1}{1-\alpha}}(1-\alpha)t)^{\frac{1}{1-\alpha}}$.

The small-noise problem for non-Lipschitz equations was studied from the different points of view, see [8], [10] and references therein. As for the study of the small-noise problem for SDE's with discontinuous multidimensional drift vector fields we also refer to [7], [2] and [14].

We propose a new approach. We make the following transformation of space and time

$$\tilde{X}(t) := \tilde{X}_\varepsilon(t) := \varepsilon^{\frac{1}{(1-\alpha)\beta-1}} X_\varepsilon(\varepsilon^{\frac{\alpha-1}{(1-\alpha)\beta-1}} t), \quad t \geq 0,$$

then we use the self similarity property of the process B_β , and show that \tilde{X} satisfies SDE

$$\tilde{X}(t) = \int_0^t (c_+ \mathbb{I}_{\tilde{X}(s) \geq 0} - c_- \mathbb{I}_{\tilde{X}(s) < 0}) \text{sign}(\tilde{X}(s)) |\tilde{X}(s)|^\alpha ds + \tilde{B}_\beta(t), \quad t \geq 0, \quad (3)$$

where $\tilde{B}_\beta \stackrel{d}{=} B_\beta$. Observe that if (3) has a unique solution, then its distribution is independent of ε . Hence, the limit behavior of X_ε as $\varepsilon \rightarrow 0$ is closely related with the behavior of $\tilde{X}(t)$ as $t \rightarrow \infty$.

Under general conditions a self-similar process with parameter β a.s. has a growth that does not exceed $t^{\beta+\delta}$ as $t \rightarrow \infty$ for any $\delta > 0$ (see [11, 12, 15]). So, if $\alpha + \beta > 1$, then $\tilde{B}_\beta(t) = o(|X^\pm(t)|), t \rightarrow \infty$, where $X^\pm = \pm(c_\pm^{\frac{1}{1-\alpha}}(1-\alpha)t)^{\frac{1}{1-\alpha}}$. Therefore it is natural to expect that if $\tilde{X}(t)$ converges to $+\infty$ (or $-\infty$) as $t \rightarrow \infty$ and $\alpha + \beta^{-1} > 1$, then $\tilde{X}(t) \sim X^+(t), t \rightarrow \infty$ (or $\sim X^-(t)$ respectively). We prove such result on asymptotic behavior of (small deterministic) perturbation of an integral equation in §3. It should be noted that assumption $\alpha + \beta^{-1} > 1$ ensures existence of a unique solution if B_β is a Lévy process [16]. Moreover uniqueness fails [16] if $\alpha + \beta^{-1} < 1$.

The application of deterministic results of §3 to the study of exact growth rate of solutions to the SDE (3) is given in §4. This question is interesting by itself. It also may have an application in the construction of consistent estimators for parameters α or c_{\pm} of a solution of equation (3). The first results on exact growth rates of solutions to stochastic differential equations were obtained by Gikhman and Skorokhod [3] who considered SDEs with Wiener noise, see also [4, 5]. We cover some of their results, and our method does not use the Ito formula. Note that cases $\alpha > 0$ and $\alpha < 0$ are considered separately (here and also in [3]). This is related with monotonicity properties of the function x^{α} and hence with the properties of the limit ODE.

The problem whether $|\tilde{X}(t)| \rightarrow \infty$ as $t \rightarrow \infty$ a.s. is significant, it also appeared in [3]. We prove almost sure convergence to infinity in the case when B_{β} is a Lévy β -stable process in §4, see Example 1.

The small-noise problem for Lévy β -stable process was solved by Flandoli and Hoeghele [9] who even find limit probabilities p_{\pm} . Their method was based on very careful study of jumps of the Lévy process and exits from the neighborhood of 0.

Our approach allows us to study the small-noise problem even in the case when $X_{\varepsilon}(0) = x_{\varepsilon} \rightarrow 0, \varepsilon \rightarrow 0$. Under suitable normalization, the corresponding limit probabilities p_{\pm} may be different from [9]. Our approach may be useful in the study of the small-noise problem where a drift behaves at 0 “similarly” to $(c_{+}\mathbb{1}_{x \geq 0} - c_{-}\mathbb{1}_{x < 0})|x^{\alpha}| \operatorname{sign} x$, or a Lévy measure of a Lévy process B is “close” to Lévy measure of some Lévy β -stable process, or even more, if the noise is not additive.

1 Main Result

Let $B_{\beta}(t), t \geq 0$ be a self-similar process with index $\beta > 0$, i.e., for any $a > 0$ the distribution of processes $\{B_{\beta}(at), t \geq 0\}$ and $\{a^{\beta}B_{\beta}(t), t \geq 0\}$ are equal. We will always assume that all considered processes have cadlag trajectories.

Consider stochastic equation

$$X_{\varepsilon}(t) = \int_0^t (c_{+}\mathbb{1}_{X_{\varepsilon}(s) \geq 0} - c_{-}\mathbb{1}_{X_{\varepsilon}(s) < 0}) \operatorname{sign}(X_{\varepsilon}(s))|X_{\varepsilon}(s)|^{\alpha} ds + \varepsilon B_{\beta}(t), \quad (4)$$

where $c_{\pm} > 0$.

Theorem 1.1. *Let $\alpha + \beta^{-1} > 1$. Assume that*

1. there exists a unique weak solution to the equation

$$X(t) = \int_0^t (c_+ \mathbb{I}_{X(s) \geq 0} - c_- \mathbb{I}_{X(s) < 0}) \operatorname{sign}(X(s)) |X(s)|^\alpha ds + B_\beta(t), \quad t \geq 0; \quad (5)$$

2. $|X(t)| \rightarrow \infty, t \rightarrow \infty$ a.s.;

3. $\alpha \in (0, 1)$ and there exists a function $h(t) = o(t^{\frac{1}{1-\alpha}}), t \rightarrow \infty$, and a random sequence $\{t_n\}, \lim_{n \rightarrow \infty} t_n = \infty$, such that $|B_\beta(t) - B_\beta(t_n)| \leq h(t - t_n), t \geq t_n$ a.s.

or

$$\alpha \in (-1, 0] \text{ and } B_\beta(t) = o(t^{\frac{1}{1-\alpha}}), t \rightarrow \infty \text{ a.s.}$$

Then the sequence of distributions of processes $\{X_\varepsilon\}$ converges in $D([0, \infty))$ to the distribution of the process

$$((1 - \alpha)t)^{\frac{1}{1-\alpha}} \left(c_+^{\frac{1}{1-\alpha}} \mathbb{I}_{\{\lim_{t \rightarrow \infty} X(t) = +\infty\}} - c_-^{\frac{1}{1-\alpha}} \mathbb{I}_{\{\lim_{t \rightarrow \infty} X(t) = -\infty\}} \right).$$

Remark 1.1. The limit process with probabilities

$$p_\pm = P(\lim_{t \rightarrow \infty} X(t) = \pm\infty)$$

equals the maximal positive (resp. the minimal negative) solution to the ODE $y'(t) = c_+ y^\alpha(t), t \geq 0, y(0) = 0$, (resp. $y'(t) = -c_- (-y)^\alpha(t)$).

2 Transformations of the main equation

Let $B_\beta(t), t \geq 0$ be a self-similar process with index $\beta > 0$.

By $X_{x,c,\alpha,\varepsilon}(t)$ denote a solution of

$$X(t) = x + \int_0^t (c_+ \mathbb{I}_{X(s) \geq 0} - c_- \mathbb{I}_{X(s) < 0}) \operatorname{sign}(X(s)) |X(s)|^\alpha ds + \varepsilon B_\beta(t). \quad (6)$$

Lemma 2.1. *Let $X_{x,c,\alpha,\varepsilon}(t)$ be a solution of (6). Then a process $\tilde{X}(t) := \varepsilon^\delta X(\varepsilon^{-\gamma} t)$ is a solution of*

$$\tilde{X}(t) = x\varepsilon^\delta + \varepsilon^{\delta(1-\alpha)-\gamma} \int_0^t (c_+ \mathbb{I}_{\tilde{X}(s) \geq 0} - c_- \mathbb{I}_{\tilde{X}(s) < 0}) \operatorname{sign}(\tilde{X}(s)) |\tilde{X}(s)|^\alpha ds +$$

$$\varepsilon^{1+\delta-\gamma\beta}\tilde{B}_\beta(t), t \geq 0, \quad (7)$$

where $\tilde{B}_\beta \stackrel{d}{=} B_\beta$.

That is, the distributions of

$$\varepsilon^\delta X_{x,c,\alpha,\varepsilon}(\varepsilon^{-\gamma}t), t \geq 0, \text{ and } X_{x\varepsilon^\delta, c\varepsilon^{\delta(\alpha-1)+\gamma}, \alpha, \varepsilon^{-\delta+\gamma\beta}}(t), t \geq 0,$$

are equal if at least one of equations (6) or (7) has a unique weak solution.

In particular, if $x = 0$, $\delta = \frac{1}{(1-\alpha)\beta-1}$, $\gamma = \frac{1-\alpha}{(1-\alpha)\beta-1}$, then $\tilde{X}(t)$ satisfies the equation (3).

Proof.

$$\begin{aligned} \tilde{X}(t) &= \varepsilon^\delta X(\varepsilon^{-\gamma}t) = x\varepsilon^\delta + \varepsilon^\delta \int_0^{\varepsilon^{-\gamma}t} (c_+ \mathbb{I}_{X(s) \geq 0} - c_- \mathbb{I}_{X(s) < 0}) \text{sign}(X(s)) |X(s)|^\alpha ds + \\ &\quad \varepsilon^\delta \varepsilon B_\beta(\varepsilon^{-\gamma}t) = \\ &\quad x\varepsilon^\delta + \varepsilon^\delta \int_0^t (c_+ \mathbb{I}_{X(z\varepsilon^{-\gamma}) \geq 0} - c_- \mathbb{I}_{X(z\varepsilon^{-\gamma}) < 0}) \text{sign}(X(z\varepsilon^{-\gamma})) |X(z\varepsilon^{-\gamma})|^\alpha dz \varepsilon^{-\gamma} + \\ &\quad + \varepsilon^\delta \varepsilon \varepsilon^{-\gamma\beta} \varepsilon^{\gamma\beta} B_\beta(\varepsilon^{-\gamma}t) = \\ &\quad x\varepsilon^\delta + \varepsilon^\delta \varepsilon^{-\gamma} \varepsilon^{-\delta\alpha} \int_0^t (c_+ \mathbb{I}_{\tilde{X}(z) \geq 0} - c_- \mathbb{I}_{\tilde{X}(z) < 0}) \text{sign}(\tilde{X}(z)) |\tilde{X}(z)|^\alpha dz + \\ &\quad + \varepsilon^{1+\delta-\gamma\beta} \tilde{B}_\beta(t), \end{aligned}$$

where $\tilde{B}_\beta(t) = \varepsilon^{\gamma\beta} B_\beta(\varepsilon^{-\gamma}t)$. □

Corollary 2.1. *Let X be a solution of (5), X_ε be a solution of (4). Then*

$$\{X_\varepsilon(t), t \geq 0\} \stackrel{d}{=} \{\varepsilon^{\frac{-1}{(1-\alpha)\beta-1}} X(\varepsilon^{\frac{1-\alpha}{(1-\alpha)\beta-1}} t), t \geq 0\}$$

if at least one of equations (5) or (4) has a unique weak solution.

3 Asymptotic behavior of perturbed integral equation

Let $c > 0, x > 0$. In this section we find sufficient conditions ensuring that a solution to the integral equation

$$y(t) = x + c \int_0^t |y(s)|^\alpha ds + g(t) \quad (8)$$

is equivalent as $t \rightarrow \infty$ to the solution of

$$z(t) = x + c \int_0^t |z(s)|^\alpha ds$$

if the function g has comparatively small growth on infinity. I.e., we will prove that

$$y(t) \sim (c(1 - \alpha)t)^{\frac{1}{1-\alpha}}, \quad t \rightarrow \infty. \quad (9)$$

Let X be a solution of (5). Under very general assumptions (see [11, 12, 15]) self-similar process B_β satisfies the following

$$\forall \delta > 0 \quad \lim_{t \rightarrow \infty} \frac{B_\beta(t)}{t^{\beta+\delta}} = 0 \quad \text{a.s.}$$

So, it is natural to expect that if ω is such that $\lim_{t \rightarrow \infty} X(t) = +\infty$ (or $= -\infty$), then $X(t) \sim (c_+(1 - \alpha)t)^{\frac{1}{1-\alpha}}$, $t \rightarrow \infty$ (or $X(t) \sim -(c_-(1 - \alpha)t)^{\frac{1}{1-\alpha}}$, respectively). We apply deterministic result on equivalence (9) to the stochastic equation (4) in the next section.

We consider the cases $\alpha \in (0, 1)$ and $\alpha \in (-1, 0]$ separately. In this cases the function x^α is increasing or decreasing, respectively. We will always assume that all functions below are measurable and locally bounded.

3.1 Case $\alpha \in (0, 1)$

In order to study the asymptotic behavior of $y(t)$ we need few simple auxiliary lemmas.

Lemma 3.1. *Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing, locally Lipschitz function of linear growth, $g_1, g_2 : [0, T] \rightarrow \mathbb{R}$ be bounded measurable functions,*

$$y_i(t) = x_i + \int_0^t a(y_i(s))ds + g_i(t), \quad t \in [0, T], \quad i = 1, 2.$$

Assume that $x_1 \leq x_2, g_1(t) \leq g_2(t), t \in [0, T]$. Then $y_1(t) \leq y_2(t), t \in [0, T]$.

Assumptions of the lemma yield that the solutions are unique and can be obtained by iterations. The corresponding inequality obviously is satisfied for iterations, so it is satisfied for the limits also.

Remark 3.1. The assumption on monotonicity of a cannot be omitted.

As a corollary of Lemma 3.1 we get the following result.

Lemma 3.2. *Let $y_1(t) \geq \varepsilon > 0, t \in [0, T]$. Assume that $y_1(t), y_2(t), t \in [0, T]$ are such that*

$$y_1(t) \leq x_1 + c \int_0^t y_1^\alpha(s) ds + g_1(t), t \in [0, T],$$

and

$$y_2(t) \geq x_2 + c \int_0^t y_2^\alpha(s) ds + g_2(t), t \in [0, T], \quad i = 1, 2,$$

where $\alpha \in (0, 1)$, $g_1, g_2 : [0, T] \rightarrow \mathbb{R}$ are bounded measurable functions.

If $x_1 \leq x_2, g_1(t) \leq g_2(t), t \in [0, T]$, then $y_1(t) \leq y_2(t), t \in [0, T]$.

Theorem 3.1. *Assume that $y(t)$ satisfies the equation*

$$y(t) = x + c \int_{t_0}^t |y(s)|^\alpha ds + g(t), t \geq 0,$$

where $c > 0, \alpha \in (0, 1), g : [0, \infty) \rightarrow \mathbb{R}$ is a measurable locally bounded function.

Assume that $\lim_{t \rightarrow \infty} y(t) = +\infty$ and there exists a sequence $\{t_n\}$, $\lim_{n \rightarrow \infty} t_n = \infty$ and a function $h(t) = o(t^{\frac{1}{1-\alpha}})$, $t \rightarrow \infty$, such that

$$|g(t) - g(t_n)| \leq h(t - t_n), \quad t \geq t_n. \quad (10)$$

Then

$$y(t) \sim (c(1 - \alpha)t)^{\frac{1}{1-\alpha}}, t \rightarrow \infty.$$

Remark 3.2. Assumption $\lim_{t \rightarrow \infty} y(t) = +\infty$ is essential, otherwise $y(t) = 0, t \geq 0$, satisfies the equation with $x = 0, g \equiv 0$. The condition $|g(t) - g(t_n)| \leq h(t - t_n), t \geq t_n$ also cannot be replaced by $g(t) = o(t^{\frac{1}{1-\alpha}}), t \rightarrow \infty$. Indeed, let $\gamma \in (0, \frac{1}{1-\alpha}), y(t) = t^\gamma, t_0 = 0, x = 0$. Then

$$\begin{aligned} y(t) &= \int_0^t (s^\gamma)^\alpha ds - \int_0^t (s^\gamma)^\alpha ds + t^\gamma = \\ &= \int_0^t |y(s)|^\alpha ds - \frac{t^{\alpha\gamma+1}}{\alpha\gamma+1} + t^\gamma = \int_0^t |y(s)|^\alpha ds + g(t), \end{aligned}$$

where $g(t) = -\frac{t^{\alpha\gamma+1}}{\alpha\gamma+1} + t^\gamma$.

Since $\alpha\gamma + 1 < \frac{\alpha}{1-\alpha} + 1 = \frac{1}{1-\alpha}$, we have $g(t) = o(t^{\frac{1}{1-\alpha}})$ and $f(t) = o(t^{\frac{1}{1-\alpha}})$ as $t \rightarrow \infty$.

Proof of Theorem 3.1. We have

$$y(t) = y(t_n) + c \int_{t_n}^t |y(s)|^\alpha ds + g(t) - g(t_n) \geq y(t_n) + c \int_{t_n}^t |y(s)|^\alpha ds + h(t - t_n), t \geq t_n.$$

Set $z_n(t) = y(t + t_n)$, $x_n = y(t_n)$. Then

$$z_n(t) \geq x_n + c \int_0^t |z_n(s)|^\alpha ds - h(t), t \geq 0. \quad (11)$$

Denote $\tilde{z}^{(a)}(t) = (1 + a(1 - \alpha)t)^{\frac{1}{1-\alpha}}$; $\tilde{z}^{(a)}$ satisfies the equation

$$\tilde{z}^{(a)}(t) = 1 + a \int_0^t |\tilde{z}^{(a)}(s)|^\alpha ds.$$

Let $c_- < c$. Then

$$\begin{aligned} \tilde{z}^{(c_-)}(t) &= 1 + c_- \int_0^t |\tilde{z}^{(c_-)}(s)|^\alpha ds = \\ &= 1 + c \int_0^t |\tilde{z}^{(c_-)}(s)|^\alpha ds + (c_- - c) \int_0^t |\tilde{z}^{(c_-)}(s)|^\alpha ds = \\ &= 1 + c \int_0^t |\tilde{z}^{(c_-)}(s)|^\alpha ds + (c_- - c) \left((1 + c_-(1 - \alpha)t)^{\frac{1}{1-\alpha}} - 1 \right) \leq \\ &\leq K(c_-) + c \int_0^t |\tilde{z}^{(c_-)}(s)|^\alpha ds - h(t), t \geq 0, \end{aligned} \quad (12)$$

where $K(c_-)$ is a constant.

It follows from (11), (12), and Lemma 3.2 that if n is sufficiently large, then

$$\overline{\lim}_{t \rightarrow \infty} \frac{z_n(t)}{\tilde{z}^{(c_-)}(t)} = \overline{\lim}_{t \rightarrow \infty} \frac{z_n(t)}{(1 + c_-(1 - \alpha)t)^{\frac{1}{1-\alpha}}} \geq 1.$$

Therefore

$$\forall c_- < c \quad \overline{\lim}_{t \rightarrow \infty} \frac{y(t)}{(c(1 - \alpha)t)^{\frac{1}{1-\alpha}}} \geq \frac{c_-^{\frac{1}{1-\alpha}}}{c^{\frac{1}{1-\alpha}}}.$$

Similarly we get the inequality

$$\forall c_+ > c \quad \underline{\lim}_{t \rightarrow \infty} \frac{y(t)}{(c(1 - \alpha)t)^{\frac{1}{1-\alpha}}} \leq \frac{c_+^{\frac{1}{1-\alpha}}}{c^{\frac{1}{1-\alpha}}}.$$

Since $c_- < c$ and $c_+ > c$ were arbitrary, this proves the theorem. \square

3.2 Case $\alpha \in (-1, 0]$

In this subsection we assume that solution of (8) is positive for all $t \geq 0$.

Lemma 3.3. *Let $\varepsilon \in (0, 1)$, $a > 1$, $\Delta \in (0, \frac{1-\alpha}{\alpha}) \cap (0, \frac{1}{\varepsilon} - 1)$ be arbitrary. There exists $\delta = \delta(\varepsilon, a, \Delta) > 0$ such that if*

$$|g(t)| \leq \delta t^{\frac{1}{1-\alpha}}, \quad t \in [a^n, a^{n+1}],$$

for some $n \geq 1$ and y is a solution of (8) such that

$$(1 - \varepsilon)(c(1 - \alpha)a^n)^{\frac{1}{1-\alpha}} < y(t) < (1 + \varepsilon)(c(1 - \alpha)a^n)^{\frac{1}{1-\alpha}}, \quad (13)$$

then for all $t \in [a^n, a^{n+1}]$ we have

$$(1 - \varepsilon(1 + \Delta))(c(1 - \alpha)a^n)^{\frac{1}{1-\alpha}} < y(t) < (1 + \varepsilon)(c(1 - \alpha)a^{n+1})^{\frac{1}{1-\alpha}}, \quad (14)$$

and

$$(1 - \varepsilon)(c(1 - \alpha)a^{n+1})^{\frac{1}{1-\alpha}} < y(a^{n+1}). \quad (15)$$

Proof. The function x^α , $x > 0$ is decreasing. So, if

$$0 < y_-(t) < y(t) < y_+(t), \quad t \in [a^n, a^{n+1}],$$

then for all $t \in [a^n, a^{n+1}]$

$$\begin{aligned} y(t) &= y(a^n) + \int_{a^n}^t cy^\alpha(s)ds + g(t) - g(a^n) < \\ &y_+(a^n) + \int_{a^n}^t cy_-^\alpha(s)ds + |g(t)| + |g(a^n)| \leq \\ &y_+(a^n) + c \max_{s \in [a^n, t]} y_-^\alpha(s)(t - a^n) + 2\delta t^{\frac{1}{1-\alpha}} \leq \\ &y_+(a^n) + ca^n(a - 1) \max_{s \in [a^n, a^{n+1}]} y_-^\alpha(s) + 2\delta a^{\frac{n+1}{1-\alpha}}. \end{aligned} \quad (16)$$

Similarly

$$y(t) > y_-(a^n) + ca^n(a - 1) \min_{s \in [a^n, a^{n+1}]} y_+^\alpha(s) - 2\delta a^{\frac{n+1}{1-\alpha}}. \quad (17)$$

By y_- and y_+ denote the left hand side and the right hand side of (14), correspondingly. To prove the lemma it suffices to show that there exists

$\delta > 0$ such that the upper (and the lower) bound of y from inequalities (16) (respectively (17)) is less than the right hand side of (14) (is greater than the left hand side of (14) and (15)).

Let us check the upper bound only. The lower bound can be proved similarly.

Let $t \in [a^n, a^{n+1}]$. We have to verify that

$$(1+\varepsilon) (c(1-\alpha)a^n)^{\frac{1}{1-\alpha}} + ca^n(a-1) \left((1-\varepsilon(1+\Delta))(c(1-\alpha)a^n)^{\frac{1}{1-\alpha}} \right)^\alpha + 2\delta a^{\frac{n+1}{1-\alpha}} < (1+\varepsilon) (c(1-\alpha)a^{n+1})^{\frac{1}{1-\alpha}}.$$

Let us move the first term to the right side and divide both sides by $(c(1-\alpha)a^n)^{\frac{1}{1-\alpha}}$. We get

$$\frac{(1-\varepsilon(1+\Delta))^\alpha(a-1)}{1-\alpha} + \frac{2\delta a^{\frac{1}{1-\alpha}}}{(c(1-\alpha))^{\frac{1}{1-\alpha}}} < (1+\varepsilon)(a^{\frac{1}{1-\alpha}} - 1). \quad (18)$$

Since $\alpha \in (-1, 0]$, the mean value theorem yields inequalities

$$\forall x \in (0, 1) \quad (1-x)^\alpha \leq 1-\alpha x; \quad \forall x > 0 \quad (1+x)^{\frac{1}{1-\alpha}} \geq 1 + \frac{x}{1-\alpha}. \quad (19)$$

It follows from (19) that to prove (18) it is sufficient to show

$$\frac{(1-\varepsilon(1+\Delta)\alpha)(a-1)}{1-\alpha} + \frac{2\delta a^{\frac{1}{1-\alpha}}}{(c(1-\alpha))^{\frac{1}{1-\alpha}}} < \frac{(1+\varepsilon)(a-1)}{1-\alpha}. \quad (20)$$

Since $(1+\Delta)\alpha < 1$ by the assumptions of the lemma, inequality (20) is true for sufficiently small δ . Lemma 3.3 is proved. \square

The following statement is a simple corollary of Lemma 3.3.

Lemma 3.4. *Assume that conditions of Lemma 3.3 are satisfied for some $\varepsilon, a, \Delta, \delta$; inequality (13) is true for some n , and*

$$|g(t)| \leq \delta t^{\frac{1}{1-\alpha}}, \quad t \geq a^n.$$

Then

$$(1-\varepsilon(1+\Delta)) \leq \liminf_{t \rightarrow \infty} \frac{y(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq \limsup_{t \rightarrow \infty} \frac{y(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq (1+\varepsilon)a^{\frac{1}{1-\alpha}}. \quad (21)$$

Theorem 3.2. Assume that $\lim_{t \rightarrow \infty} \frac{g(t)}{t^{\frac{1}{1-\alpha}}} = 0$. Assume that the solution of (8) is such that

$$\liminf_{t \rightarrow \infty} y(t) > 0.$$

Then

$$\lim_{t \rightarrow \infty} \frac{y(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} = 1.$$

Proof. Let $\varepsilon > 0, \Delta \in (0, \frac{1-\alpha}{\alpha}) \cap (0, \frac{1}{\varepsilon} - 1), a > 1$ be arbitrary. Select $\delta > 0$ from Lemma 3.3 and n such that

$$\inf_{t \geq a^n} y(t) > 0, \quad \sup_{t \geq a^n} \frac{|B(t)|}{t^{\frac{1}{1-\alpha}}} \leq \delta.$$

Let

$$x(t) = (c(1-\alpha)a^n)^{\frac{1}{1-\alpha}} + c \int_{a^n}^t x^\alpha(s) ds + B(t) - B(a^n).$$

Note that $x(t) > 0, t \geq a^n$ by Lemma 3.3.

It follows from Lemma 3.4 that

$$(1 - \varepsilon(1 + \Delta)) \leq \liminf_{t \rightarrow \infty} \frac{x(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq \limsup_{t \rightarrow \infty} \frac{x(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq (1 + \varepsilon)a^{\frac{1}{1-\alpha}}.$$

Assume that $y(a^n) \geq x(a^n)$. Then by the comparison theorem $y(t) \geq x(t) > 0, t \geq a^n$. Since $\alpha < 0$, we have

$$x(t) - y(t) \leq x(a^n) - y(a^n) + \int_{a^n}^t (x^\alpha(s) - y^\alpha(s)) ds \leq x(a^n) - y(a^n), t \geq a^n.$$

Similarly, if $y(a^n) \leq x(a^n)$, then $x(t) - y(t) \geq x(a^n) - y(a^n), t \geq a^n$. Hence, in any case

$$|x(t) - y(t)| \leq |x(a^n) - y(a^n)|, t \geq a^n.$$

Therefore

$$(1 - \varepsilon(1 + \Delta)) \leq \liminf_{t \rightarrow \infty} \frac{y(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq \limsup_{t \rightarrow \infty} \frac{y(t)}{(c(1-\alpha)t)^{\frac{1}{1-\alpha}}} \leq (1 + \varepsilon)a^{\frac{1}{1-\alpha}}.$$

Since $\varepsilon > 0, \Delta > 0$, and $a > 1$ were arbitrary, the theorem is proved. \square

4 Proof of the main results

The following statements is a corollary of Theorem 3.1 and 3.2.

Theorem 4.1. *Let $\tilde{X}(t), t \geq 0$, be a solution to SDE*

$$\tilde{X}(t) = \int_0^t (c_+ \mathbb{I}_{\tilde{X}(s) \geq 0} - c_- \mathbb{I}_{\tilde{X}(s) < 0}) \text{sign}(\tilde{X}(s)) |\tilde{X}(s)|^\alpha ds + B(t), t \geq 0,$$

where $c_\pm > 0$, B is a measurable stochastic process.

Suppose that

1) $\alpha \in (0, 1)$ and there exists a function $h(t) = o(t^{\frac{1}{1-\alpha}}), t \rightarrow \infty$, and a (random) sequence $\{t_n\}$, $\lim_{n \rightarrow \infty} t_n = \infty$, such that

$$|B(t) - B(t_n)| \leq h(t - t_n), t \geq t_n \text{ a.s.}$$

or

2) $\alpha \in (-1, 0]$ and $B(t) = o(t^{\frac{1}{1-\alpha}}), t \rightarrow \infty$ a.s.

Then

$$\begin{aligned} \tilde{X}(t) &\sim (c_+(1-\alpha)t)^{\frac{1}{1-\alpha}} \text{ for a.a. } \omega \in \{\lim_{t \rightarrow \infty} \tilde{X}(t) = +\infty\}, \\ \tilde{X}(t) &\sim -(c_-(1-\alpha)t)^{\frac{1}{1-\alpha}} \text{ for a.a. } \omega \in \{\lim_{t \rightarrow \infty} \tilde{X}(t) = -\infty\}. \end{aligned}$$

To solve the selection problem for a process X_ε defined in (4) we need the following result.

Lemma 4.1. *Assume that function f is such that $f(t) \sim Kt^A, t \rightarrow \infty$, where $A > 0$.*

Set $g_n(t) = n^{-1}f(n^{1/A}t)$. Then for any $[a, b] \subset (0, \infty)$ we have

$$\lim_{n \rightarrow \infty} \sup_{s \in [a, b]} |g_n(s) - Ks^A| = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Select t_0 such that

$$\left| \frac{f(t)}{Kt^A} - 1 \right| < \varepsilon, t \geq t_0.$$

Then for sufficiently large n :

$$(Kb^A)^{-1} \sup_{s \in [a, b]} |g_n(s) - Ks^A| \leq \sup_{s \in [a, b]} \left| \frac{g_n(s)}{Ks^A} - 1 \right| = \sup_{s \in [a, b]} \left| \frac{f(n^{1/A}s)}{K(n^{1/A}s)^A} - 1 \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the lemma is proved. \square

Let X be a solution to (5). It follows from uniqueness of a solution and Corollary 2.1 that the distribution of $X_\varepsilon(t), t \geq 0$, equals the distribution of $\tilde{X}_\varepsilon(t) := \varepsilon^{\frac{-1}{(1-\alpha)\beta-1}} X(\varepsilon^{\frac{1-\alpha}{(1-\alpha)\beta-1}} t), t \geq 0$.

Let ω be such that $\lim_{t \rightarrow \infty} X(t) = +\infty$. Then due to Theorems 3.1 and 3.2, we have $X(t) \sim (c_+(1-\alpha)t)^{\frac{1}{1-\alpha}}, t \rightarrow \infty$ for this ω .

Set $f = X, n = \varepsilon^{\frac{1}{(1-\alpha)\beta-1}}, A = \frac{1}{1-\alpha}, K = (c_+(1-\alpha))^{\frac{1}{1-\alpha}}$ in Lemma 4.1. Then for any $0 < a < b$ we get $\lim_{\varepsilon \rightarrow 0+} \sup_{s \in [a,b]} |\tilde{X}_\varepsilon(t) - (c_+(1-\alpha)t)^{\frac{1}{1-\alpha}}| = 0$.

Similarly we obtain the corresponding uniform convergence for ω such that $\lim_{t \rightarrow \infty} X(t) = -\infty$.

Observe that $\sup_{t \in [0,\delta]} |\tilde{X}_\varepsilon(t)|$ does not exceed $Y_\varepsilon(\delta)$, where Y_ε is a solution of the following equation

$$Y_\varepsilon(t) = 2\varepsilon \sup_{s \in [0,\delta]} |B(s)| + (c_+ + c_-) \int_0^t |Y_\varepsilon(s)|^\alpha ds, \quad t \in [0, \delta].$$

To conclude the proof of Theorem 1.1 notice that

$$\sup_{t \in [0,\delta]} \sup_{\varepsilon \in [0,1]} |\tilde{X}_\varepsilon(t)| \leq \sup_{\varepsilon \in [0,1]} |Y_\varepsilon(\delta)| = |Y_1(\delta)| \xrightarrow{P} 0, \delta \rightarrow 0+.$$

Example 1 (Symmetric Lévy stable noise). Let $B_\beta(t), t \geq 0$ be a symmetric $1/\beta$ -stable process. Then B_β is self-similar process with parameter β .

It follows from [16] that there exists a unique (strong) solution to (3) if $\alpha + 1/\beta > 1$. This solution is a strong Markov process.

Remark 4.1. We give proofs only for $\alpha \in (0, 1)$. The case $\alpha \in (-1, 0]$ is simpler.

Let $\varepsilon > 0$ be such that $\alpha + (\beta + \varepsilon)^{-1} > 1$. Set $h(t) = 1 + t^{\beta+\varepsilon}$.

It follows from [11] that $\lim_{t \rightarrow \infty} B_\beta(t)/h(t) = 0$ a.s.

It is easy to see that

$$\gamma := P(|B_\beta(t)| > h(t), t \geq 0) > 0.$$

Let us verify that

$$\lim_{t \rightarrow \infty} |\tilde{X}(t)| = \infty \quad \text{a.s.} \quad (22)$$

To prove this it suffices to show that

$$\forall K > 0 \quad P\left(\exists t_0 \forall t \geq t_0 \quad |\tilde{X}(t)| \geq K\right) = 1. \quad (23)$$

Let K be fixed. It follows from the results of §3 that there exists $M = M(K)$ such that

if ω and t_0 are such that $|\tilde{X}(t_0)| \geq M$ and $|B_\beta(s+t_0) - B_\beta(t_0)| > h(s)$, $s \geq 0$, then $|\tilde{X}(t)| \geq K$, $t \geq t_0$.

Set

$$\tau_1 := \inf\{t \geq 0 : |\tilde{X}(t)| \leq K\}, \quad \sigma_n := \inf\{t \in \mathbb{N} \cap [\tau_n, \infty) : |\tilde{X}(t)| \geq M\},$$

$$\tau_{n+1} := \inf\{t \geq \sigma_n : |\tilde{X}(t)| \leq K\}, \quad n \geq 1.$$

Then by the strong Markov property,

$$X_n := \begin{cases} \tilde{X}(\tau_n), & \tau_n < \infty, \\ \infty, & \tau_n = \infty. \end{cases}$$

is a homogeneous Markov chain.

It is obvious that $P(\sigma_n < \infty | \tau_n < \infty) = 1$. Hence

$$\sup_{|x| \leq K} P_x(X_1 = \infty | X_0 = x) \geq P(|B_\beta(s + \sigma_1) - B_\beta(\sigma_1)| > h(s), s \geq 0) = \gamma > 0.$$

Therefore

$$P(X_n < \infty) \leq (1 - \gamma)^n.$$

So

$$P(\exists n_0 : X_{n_0} = \infty) = 1.$$

This yields (23) and hence (22).

Thus we have proved that all conditions of Theorems 4.1 and 1.1 are satisfied for $1/\beta$ -stable Lévy processes if $\alpha + 1/\beta > 1$. Therefore for a.a. ω we have either $\tilde{X}(t) \sim (c_+(1-\alpha)t)^{\frac{1}{1-\alpha}}$ or $\tilde{X}(t) \sim -(c_-(1-\alpha)t)^{\frac{1}{1-\alpha}}$ as $t \rightarrow \infty$. Moreover X_ε converges in distribution as $\varepsilon \rightarrow 0$ to

$$((1-\alpha)t)^{\frac{1}{1-\alpha}} \left(c_+^{\frac{1}{1-\alpha}} \mathbb{I}_{\{\lim_{t \rightarrow \infty} \tilde{X}(t) = +\infty\}} - c_-^{\frac{1}{1-\alpha}} \mathbb{I}_{\{\lim_{t \rightarrow \infty} \tilde{X}(t) = -\infty\}} \right).$$

References

- [1] BAFICO, R., BALDI, P. (1982). Small random perturbations of Peano phenomena. *Stochastics*, **6**, n. 3, 279-292.
- [2] BUCKDAHN, R., OUKNINE, Y., QUINCAMPOIX, M. (2009). On limiting values of stochastic differential equations with small noise intensity tending to zero. *Bull. Sci. Math.* **133:3**, 229-237.

- [3] GIKHMAN, I.I., SKOROKHOD, A.V. Stochastic differential equations. (Russian) Naukova Dumka, Kiev. 1968. - 354 p.; English translation - Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72. - Springer-Verlag. New York; Heidelberg. 1972. - viii+354 p.
- [4] KELLER, G., KERSTING, G. & ROSLER, U. (1984) On the asymptotic behaviour of solutions of stochastic differential equations, *Z. Wahrsch. Verw. Geb.*, **68:2**, 163–184.
- [5] BULDYGIN, V. V., KLESOV, O. I., STEINEBACH, J. G. & TY-MOSHENKO, O. A. (2008) On the φ -asymptotic behaviour of solutions of stochastic differential equations. *Theory of Stochastic Processes*, **14 (30)**, no. 1, 11–29.
- [6] ENGELBERT, H. J., SCHMIDT, W. (1991) Strong Markov Continuous Local Martingales and Solutions of One-Dimensional Stochastic Differential Equations (Part III). *Mathematische Nachrichten*, **151(1)**, 149-197.
- [7] Delarue, F., Flandoli, F., Vincenzi, D. (2014). Noise prevents collapse of Vlasov-Poisson point charges, *Communications on Pure and Applied Math.*, **67**, Issue 10, 1700-1736.
- [8] FLANDOLI, F. (2010). Random Perturbation of PDEs and Fluid Dynamic Models. *Lecture Notes in Mathematics*, **2015**, Springer.
- [9] FLANDOLI, F., HOGELE, M. (2014). A solution selection problem with small symmetric stable perturbations. *arXiv preprint* , arXiv:1407.3469.
- [10] FLANDOLI, F. (2009). Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations. *Metrika*, **69. 2-3**, 101-123.
- [11] KHINTCHINE, A. (1938) Zwei Sätze über stochastische Prozesse mit stabilen Verteilungen, *Rec. Math. [Mat. Sbornik] N.S.*, **3(45):3**, 577–584.
- [12] KONO, N. (1983). Iterated log type strong limit theorems for self-similar processes. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, **59(3)**, 85-87.

- [13] NUALART, D., OUKNINE, Y. (2002). Regularization of differential equations by fractional noise. *Stochastic Processes and their Applications*, **102(1)**, 103-116.
- [14] PILIPENKO, A., PROSKE, F. (2015). On a selection problem for small noise perturbation in multidimensional case. *arXiv preprint*, arXiv:1510.00966.
- [15] TAKASHIMA, K. (1989). Sample path properties of ergodic self-similar processes. *Osaka Journal of Mathematics*, **26(1)**, P.159-P.189
- [16] TANAKA, H., TSUCHIYA, M., & WATANABE, S. (1974). Perturbation of drift-type for Lévy processes. *Journal of Mathematics of Kyoto University*, **14(1)**, 73-92.